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1997 J. Phys. A: Math. Gen. 30 6535

(http://iopscience.iop.org/0305-4470/30/18/027)

J. Phys. A: Math. Gen. **30** (1997) 6535–6543. Printed in the UK PII: S0305-4470(97)82446-8

Free motion on the Poisson $SU(N)$ group

S Zakrzewski

Department of Mathematical Methods in Physics, University of Warsaw, Hoża 74, 00-682 Warsaw, Poland

Received 11 March 1997

Abstract. $SL(N, \mathbb{C})$ is the phase space of the Poisson $SU(N)$. We explicitly calculate the symplectic structure of $SL(N, \mathbb{C})$, define an analogue of the Hamiltonian of the free motion on $SU(N)$ and solve the corresponding equations of motion. Velocity is related to the momentum by a nonlinear Legendre transformation.

1. Introduction

The theory of Poisson groups [1–5] (and their phase spaces [6–9]) allows us to consider deformations of known mechanical models. Several low-dimensional examples, related to Poisson symmetry, have already been investigated [10–15]. All these models certainly have their quantum-mechanical counterpart, the underlying Poisson group being replaceable by the corresponding quantum group. It is natural to first study the Poisson case as technically simpler. We obtain interesting classical systems, and, at the same time, we get some idea about the corresponding quantum systems.

Recall that if we agree to generalize ordinary configuration manifolds by using Poisson manifolds, the ordinary phase spaces (cotangent bundles) have to be replaced by symplectic groupoids of these Poisson manifolds [6–9] (we still call them phase spaces). If the Poisson manifold is a Poisson group, its phase space can be described quite explicitly [5, 8] (this is not true in the general case). Namely, let $(\mathfrak{g}, \delta)$ be the (tangent) Lie bialgebra of our Poisson group. On the Lie algebra $\mathfrak{m}$ of the corresponding Manin triple (we call $\mathfrak{m}$ just the *double* of $(\mathfrak{g}, \delta)$), there is the Drinfeld's canonical r-matrix r_D. It defines two Poisson structures on the Lie group M corresponding to $\mathfrak{m}$:

(1) the *difference* of the right and left translation of r_D (the Sklyanin bracket). It defines a Poisson Lie group structure called the *Drinfeld double* (of the original Poisson group). This Poisson structure is always degenerate (it vanishes at the group unit). The corresponding Lie bialgebra structure on $\mathfrak{m}$ is said to be the *Drinfeld double* of $(\mathfrak{g}, \delta)$.

(2) The *sum* of the right and left translation of r_D. This Poisson structure is interesting for us because it gives the symplectic structure of the phase space of the initial Poisson group [5, 8]. We refer to M equipped with this Poisson structure as to the *Heisenberg double* (of the original Poisson group).

In this paper we calculate the explicit form of Poisson brackets on the phase space of the Poisson $SU(N)$ group, i.e. on $SL(N, \mathbb{C})$ (as a real manifold). We also consider a natural candidate for the Hamiltonian of the free motion. It turns out that the projections of the phase trajectories onto $SU(N)$ are 'big circles' (shifted one-parameter subgroups), as in the usual case. The (constant) velocity is, however, a nonlinear function of the momentum, so we have an example of a deformed Legendre transformation.

0305-4470/97/186535+09\$19.50

The case of $SU(2)$ was presented in [13], in a direct (tedious) way—without referring to the compact r-matrix notation. The above-mentioned deformed character of the Legendre transformation in this case was shown in [15] to be the reason why the free dynamics reduced to the homogeneous space (Poisson sphere) yields really a deformation of usual free trajectories on the sphere.

The paper is organized as follows. In section 2 we clarify when the Manin Lie algebra (the *double*) of a Lie bialgebra $(\mathfrak{g}, \delta)$ coincides with the complexification of $\mathfrak{g}$ (recall [4] that this is the case of $\mathfrak{g} = su(N)$), and we obtain a useful formula for the Drinfeld's canonical r-matrix r_D on the double. In section 3 we calculate r_D for $\mathfrak{g} = su(N)$ in terms of matrix units. This allows us to effectively write down the Poisson brackets of matrix elements of $SL(N, \mathbb{C})$. In section 4 we introduce the free Hamiltonian, which is one of the most natural functions on $SL(N, \mathbb{C})$. We solve the equations of motion and analyse the bijectivity property of the 'Legendre transformation'.

2. The double and complexification

Let $\mathfrak{g}$ be a real Lie algebra and let $b(\cdot, \cdot)$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. On the complexification $\mathfrak{g}^{\mathbb{C}}$ we then have the non-degenerate invariant-symmetric bilinear form $B := \operatorname{Im} b^{\mathbb{C}}$, with respect to which both $\mathfrak{g}$ and $i\mathfrak{g}$ are isotropic. We are thus almost in the situation of a Manin triple: all properties are satisfied except that $\mathrm{i}\mathfrak{g}$ is not a Lie subalgebra (unless $\mathfrak{g}$ is Abelian).

Of course, any isotropic Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}^{\mathbb{C}}$, which is complementary to $\mathfrak{g}$, yields a Manin triple and the corresponding Lie bialgebra structure on $\mathfrak{g}$. The question now arises, which Lie bialgebra structures on $\mathfrak{g}$ are obtained in this way.

A subspace $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ is complementary to $\mathfrak{g}$ if it is of the form

$$\mathfrak{h} = \{\mathrm{i}x + Rx : x \in \mathfrak{g}\} \tag{1}$$

where $R : \mathfrak{g} \to \mathfrak{g}$ is a linear map. Such a subspace is isotropic (with respect to B) if and only if R is skew-symmetric with respect to b:

$$b(Rx, y) = -b(x, Ry) \qquad x, y \in \mathfrak{g}. \tag{2}$$

This subspace is a subalgebra if and only if

$$[Rx, Ry] - R([Rx, y] + [x, Ry]) = [x, y] \qquad x, y \in \mathfrak{g}. \tag{3}$$

Let $s \in \mathfrak{g} \otimes \mathfrak{g}$ denote the inverse of b. We shall use the same letter for s considered as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$. The composition $r := Rs$ is then a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$, which we can also identify with an element of $\mathfrak{g} \otimes \mathfrak{g}$ (an element of $\mathfrak{g} \otimes \mathfrak{g}$ defines a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ by the contraction in the first argument). In terms of this $r \in \mathfrak{g} \otimes \mathfrak{g}$, condition (2) means that r is antisymmetric, and condition (3) is equivalent to

$$[[r, r]] = [[s, s]] \tag{4}$$

where $[[w, w]]$ for $w \in \mathfrak{g} \otimes \mathfrak{g}$ denotes the 'Leningrad school' bracket

$$[[w, w]] := [w_{12}, w_{13}] + [w_{12}, w_{23}] + [w_{13}, w_{23}]$$

(for antisymmetric elements, it coincides with one half of the Schouten bracket). In the terminology of [14], it means that $r + \mathrm{i}s$ is an *imaginary* quasitriangular classical r-matrix:

$$[[r + \mathrm{i}s, r + \mathrm{i}s]] = 0.$$

It is easy to show that the Lie bialgebra structure on $\mathfrak{g}$ defined by r (by taking the coboundary of r), coincides with the one defined by the Manin triple $(\mathfrak{g}^{\mathbb{C}}; \mathfrak{g}, \mathfrak{h})$:

$$\begin{aligned} B([\mathrm{i}x + Rx, \mathrm{i}y + Ry], z) &= b([Rx, y] + [x, Ry], z) \\ &= b(y, \mathrm{ad}_z\, Rx) - b(x, \mathrm{ad}_z\, Ry) \\ &= (b \otimes b)(x \otimes y, (\mathrm{id} \otimes \mathrm{ad}_z)r) + (b \otimes b)(x \otimes y, (\mathrm{ad}_z \otimes \mathrm{id})r) \\ &= (B \otimes B)((\mathrm{i}x + Rx) \otimes (\mathrm{i}y + Ry), \mathrm{ad}_z\, r). \end{aligned}$$

We summarize this discussion in the following proposition, where $\mathfrak{g}$, b are as above and $B = \mathrm{Im}\, b^{\mathbb{C}}$, $s = b^{-1}$.

Proposition 2.1. There is a one-to-one correspondence between Manin triples $(\mathfrak{g}^{\mathbb{C}}; \mathfrak{g}, \mathfrak{h})$ realized in $\mathfrak{g}^{\mathbb{C}}$ (with the scalar product B) and imaginary quasitriangular classical r-matrices $r + \mathrm{i}s$ on $\mathfrak{g}$. The correspondence is given by (1) and $r = Rs$.

The Drinfeld double quasitriangular structure on $\mathfrak{g}^{\mathbb{C}}$ is given by the canonical element

$$w_D = e_k \otimes f^k \in \mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}} \tag{5}$$

(summation convention), where e_k is a basis of $\mathfrak{g}$ and f^j is the dual (with respect to B) basis in $\mathfrak{h}$. One can easily see that $f^j = r(e^j) + \mathrm{i}s(e^j)$, where e^j is the dual basis in $\mathfrak{g}^*$.

The skew-symmetric part r_D of w_D is given by

$$r_D = \tfrac{1}{2} e_j \wedge [r(e^j) + \mathrm{i}s(e^j)] = r + \tfrac{1}{2} e_j \wedge (\mathrm{i}s^{jk} e_k). \tag{6}$$

Note that both w_D and r_D are elements of the real tensor product $V \otimes_{\mathbb{R}} V$, where $V := \mathfrak{g}^{\mathbb{C}}$ is treated as a real vector space. They may be, however, treated as (real) elements of the complexification

$$(V \otimes_{\mathbb{R}} V)^{\mathbb{C}} = V^{\mathbb{C}} \otimes_{\mathbb{C}} V^{\mathbb{C}}$$

which is much more convenient. In order to distinguish the imaginary unit i arising in the complexification of V from the imaginary unit arising in the complexification of $\mathfrak{g}$, we denote the latter by $J : \mathfrak{g} \to \mathfrak{g}$. Recall, that any $v \in V$ may be represented as the sum

$$v = v^+ + v^- \qquad v^{\pm} := \frac{1}{2}\left(v \pm \frac{1}{\mathrm{i}} Jv\right)$$

and we have

$$(Jv)^{\pm} = \pm \mathrm{i} v^{\pm} \qquad \text{(we have also } [v_1^{\pm}, v_2^{\pm}] = \pm [v_1, v_2]^{\pm}\text{)}.$$

In a fixed basis, it is also convenient to set

$$e_j = \partial_j + \bar{\partial}_j \qquad \text{where } \partial_j := e_j^+,\ \bar{\partial}_j := e_j^-.$$

In particular, the last term in (6) may be written as follows

$$\tfrac{1}{2} s^{jk} (\partial_j + \bar{\partial}_j) \wedge (\mathrm{i}\partial_j - \mathrm{i}\bar{\partial}_k) = \mathrm{i}s^{jk} \bar{\partial}_j \wedge \partial_k.$$

This term will be denoted by $s^\wedge$. Note that

$$s^\wedge = \wedge(\mathrm{id} \otimes J)s \tag{7}$$

where $\wedge(a \otimes b) := a \wedge b$ is the antisymmetrization.

3. Drinfeld and Heisenberg double of Poisson $SU(N)$

Let b denote the invariant scalar product on $\mathfrak{g} := su(N)$ given by

$$b(X, Y) := -\frac{1}{\varepsilon}\,\mathrm{tr}\,XY \tag{8}$$

(ε is a parameter). Let $\mathfrak{h} = sb(N)$ be the Lie subalgebra in $\mathfrak{g}^{\mathbb{C}} = sl(N, \mathbb{C})$ consisting of complex upper-triangular matrices with real diagonal elements (and trace zero). It is easy to see that $\mathfrak{h}$ is complementary to $\mathfrak{g}$ and isotropic with respect to $B = \mathrm{Im}\, b^{\mathbb{C}}$, hence $(\mathfrak{g}^{\mathbb{C}}; \mathfrak{g}, \mathfrak{h})$ is a Manin triple. It corresponds to the standard Poisson $SU(N)$ [4]. Our aim is to calculate $r_D = r + s^{\wedge}$ given by (6). We first introduce the typical elements of $su(N)$ defined in terms of usual matrix units ${e_j}^k = e_j \otimes e^k$:

$${F_j}^k := {e_j}^k - {e_k}^j \qquad {G_j}^k := \mathrm{i}({e_j}^k + {e_k}^j) \qquad H_{jk} := \mathrm{i}({e_j}^j - {e_k}^k)$$

so that

$${F_j}^k,\ {G_j}^k\ (j < k) \qquad H_{j,j+1}\ (1 \leqslant j \leqslant N-1) \tag{9}$$

is a basis of $su(N)$.

Lemma 3.1. We have

$$r = \frac{\varepsilon}{2} \sum_{j<k} {F_j}^k \wedge {G_j}^k \tag{10}$$

$$s = \frac{\varepsilon}{2} \sum_{j<k} \left({F_j}^k \otimes {F_j}^k + {G_j}^k \otimes {G_j}^k + \frac{2}{N} H_{jk} \otimes H_{jk} \right). \tag{11}$$

Proof. It is easy to first calculate R defined in (1). Since $\mathrm{i}x + Rx \in sb(N)$ for $x \in su(N)$, it is easy to calculate the lower-triangular part of Rx (it is the corresponding part of $-\mathrm{i}x$) and the diagonal part of Rx (it is the diagonal part of $-\mathrm{i}x$ plus something real, hence zero). We obtain $RH_{jk} = 0$, $R{F_j}^k = {G_j}^k$, $R{G_j}^k = -{F_j}^k$. Since ${F_j}^k$, ${G_j}^k$ $(j < k)$ form an orthogonal set with

$$b({F_j}^k, {F_j}^k) = \frac{2}{\varepsilon} = b({G_j}^k, {G_j}^k)$$

and they are orthogonal to all H_{jk}, it is easy to check that contraction of the r given in (10) with the basis elements (using b) coincides with the action of R on these elements.

In order to prove (11), first note that only the last term needs to be explained (due to the orthogonality). The scalar product on the Cartan subalgebra spanned by H_{jk} is naturally the restriction of the scalar product defined on the space spanned by $H_j := \mathrm{i}{e_j}^j$ (the Cartan for $u(N)$) by the same formula:

$$b(H_j, H_k) = -\frac{1}{\varepsilon}\,\mathrm{tr}\, H_j H_k = \frac{1}{\varepsilon}\delta_{jk}.$$

In order to invert b on the subspace, it is sufficient to invert it on the bigger space, which is easy:

$$\varepsilon \sum_j H_j \otimes H_j \tag{12}$$

and project it orthogonally on the subspace. Since

$$H_j = \frac{1}{N}\sum_k H_k + \frac{1}{N}\sum_k (H_j - H_k)$$

is the orthogonal decomposition, we just have to replace H_j in (12) by $\frac{1}{N}\sum_k (H_j - H_k)$. This indeed gives (11). $\square$

Now we can express r, $s^\wedge$ and finally r_D in terms of 'holomorphic' and 'antiholomorphic' vectors $\partial_j{}^k = (e_j{}^k)^+$, $\bar\partial_j{}^k = (e_j{}^k)^-$. This will enable us to calculate the Poisson brackets of basic coordinate functions (and their complex conjugates) on $SL(N,\mathbb{C})$.

A straightforward insertion of $e_j{}^k = \partial_j{}^k + \bar\partial_j{}^k$, $Je_j{}^k = \mathrm{i}\partial_j{}^k - \mathrm{i}\bar\partial_j{}^k$ into (10) and (11) together with (7) gives

$$r = r^{(2,0)} + r^{(0,2)} + r^{(1,1)}$$

where

$$r^{(2,0)} = \overline{r^{(0,2)}} = \mathrm{i}\varepsilon \sum_{j<k} \partial_j{}^k \wedge \partial_k{}^j \qquad r^{(1,1)} = \mathrm{i}\varepsilon \sum_{j<k} (\bar\partial_j{}^k \wedge \partial_j{}^k - \bar\partial_k{}^j \wedge \partial_k{}^j) \tag{13}$$

and

$$s^\wedge = -\mathrm{i}\varepsilon \left(\frac{1}{N} \bar I \wedge I - \sum_{j,k} \bar\partial_j{}^k \wedge \partial_j{}^k \right) \tag{14}$$

where $I := \sum_k \partial_k{}^k$.

The r-matrix $r_D = r + s^\wedge$ on $sl(N,\mathbb{C})$ defines two Poisson bivector fields on $SL(N,\mathbb{C})$:

$$\pi_\pm(g) = r_D(g\otimes g) \pm (g\otimes g) r_D \qquad g \in SL(N,\mathbb{C}).$$

Drinfeld double of the Poisson $SU(N)$ is the Poisson group $(SL(N,\mathbb{C}), \pi_-)$. The *Heisenberg double* of the Poisson $SU(N)$ is the Poisson manifold $(SL(N,\mathbb{C}), \pi_+)$. It plays the role of the phase space (cotangent bundle) of the Poisson $SU(N)$. The bivector field π_+ is known to be non-degenerate ([5, 8]), because $SL(N,\mathbb{C})$ globally decomposes (by the Iwasawa decomposition) onto $G = SU(N)$ and $G^* := SB(N)$, i.e. every element $g \in SL(N,\mathbb{C})$ is a product of the form

$$g = u\beta \qquad u \in G,\ \beta \in G^*$$

with uniquely defined u, β. Here $SB(N)$ is the connected subgroup of $SL(N,\mathbb{C})$, corresponding to the Lie algebra $\mathfrak{h} = sb(N)$ (i.e. the Poisson *dual* of the Poisson $SU(N)$).

Using the compact notation

$$\{g_1, g_2\}^{ab}_{cd} = \{g^a_c, g^b_d\} \qquad (g_1 g_2)^{ab}_{cd} = (g\otimes g)^{ab}_{cd} = g^a_c g^b_d$$

we can now write the Poisson brackets of matrix elements of g for $\pi_\pm$ as follows

$$\{g_1, g_2\}_\pm = \rho g_1 g_2 \pm g_1 g_2 \rho \qquad \{\bar g_1, g_2\}_\pm = w' \bar g_1 g_2 \pm \bar g_1 g_2 w' \tag{15}$$

where $\rho := r_D^{(2,0)} = r^{(2,0)}$ is the purely holomorphic part of r_D and

$$w' := -\mathrm{i}\varepsilon \left(\frac{1}{N} \bar I \otimes I - \sum_k \bar\partial_k{}^k \otimes \partial_k{}^k - 2 \sum_{j<k} \bar\partial_j{}^k \otimes \partial_j{}^k \right)$$

is the antiholomorphic–holomorphic part (without antisymmetrization) of r_D, i.e. $r_D^{(1,1)} = r^{(1,1)} + s^\wedge = w' - w'_{21}$.

The Poisson structure on $SL(N,\mathbb{C})$ viewed as the Drinfeld double of the Poisson $SU(N)$ is therefore described by the brackets

$$\{g^j_l, g^j_m\}_- = \mathrm{i}\varepsilon g^j_l g^j_m \qquad (l<m)$$
$$\{g^j_l, g^k_l\}_- = \mathrm{i}\varepsilon g^j_l g^k_l \qquad (j<k)$$
$$\{g^j_l, g^k_m\}_- = 2\mathrm{i}\varepsilon g^j_m g^k_l \qquad (l<m,\ j<k)$$
$$\{g^j_l, g^k_m\}_- = 0 \qquad (l>m,\ j<k)$$

as far as the holomorphic variables are concerned (quite standard), and

$$\{\overline{g}_l^j, g_m^k\}_- = \mathrm{i}\varepsilon\left[\delta^{jk}\left(\overline{g}_l^j g_m^j + 2\sum_{a>j}\overline{g}_l^a g_m^a\right) - \delta_{lm}\left(\overline{g}_l^j g_l^k + 2\sum_{a<l}\overline{g}_a^j g_a^k\right)\right] \tag{16}$$

for the mixed case (this is nothing but the Poisson version of commutation relations for the real quantum group $SL(N,\mathbb{C})$, cf [17], formulae (3.77)–(3.80)).

The Heisenberg double Poisson structure on $SL(N,\mathbb{C})$ is given by

$$\{g_l^j, g_m^j\}_+ = -\mathrm{i}\varepsilon g_l^j g_m^j \qquad (l < m)$$
$$\{g_l^j, g_l^k\}_+ = \mathrm{i}\varepsilon g_l^j g_l^k \qquad (j < k)$$
$$\{g_l^j, g_m^k\}_+ = 0 \qquad (l < m,\ j < k)$$
$$\{g_l^j, g_m^k\}_+ = 2\mathrm{i}\varepsilon g_m^j g_l^k \qquad (l > m,\ j < k)$$
$$\{\overline{g}_l^j, g_m^k\}_+ = \mathrm{i}\varepsilon\left[-\frac{2}{N}\overline{g}_l^j g_m^k + \delta^{jk}\left(\overline{g}_l^j g_m^j + 2\sum_{a>j}\overline{g}_l^a g_m^a\right) + \delta_{lm}\left(\overline{g}_l^j g_l^k + 2\sum_{a<l}\overline{g}_a^j g_a^k\right)\right]. \tag{17}$$

It is sometimes convenient to replace the complex conjugate variable $\overline{g}$ by $g^\dagger$—the Hermitian conjugate of g. In this case the second equality in (15) is replaced by

$$\{g_1^\dagger, g_2\}_\pm = g_1^\dagger[(\tau\otimes\mathrm{id})w']g_2 \pm g_2[(\tau\otimes\mathrm{id})w']g_1^\dagger \tag{18}$$

where τ denotes the transposition. If we set

$$w := -(\tau\otimes\mathrm{id})w' = \mathrm{i}\varepsilon\left(\frac{1}{N}I\otimes I - \sum_k \partial_k{}^k\otimes\partial_k{}^k - 2\sum_{j<k}\partial_k{}^j\otimes\partial_j{}^k\right)$$

we can write these brackets as follows

$$\{g_1^\dagger, g_2\}_\pm = -g_1^\dagger w g_2 \mp g_2 w g_1^\dagger. \tag{19}$$

Note that the antisymmetric part $\frac{1}{2}(w - w_{21})$ of w coincides with ρ, the symmetric part equals

$$w - \rho = \frac{1}{2}(w + w_{21}) = \mathrm{i}\varepsilon\left(\frac{1}{N}I\otimes I - P\right) \tag{20}$$

where P is the permutation, and $\mathrm{i}w$ is the infinitesimal part of the R-matrix for the A_N-series (cf [18, 17]),

$$\mathcal{R} = I\otimes I + \mathrm{i}w + \ldots$$

(when $q = 1 + \varepsilon + \cdots$). In particular, w satisfies the classical Yang–Baxter equation $[[w, w]] = 0$ (this can be shown also by purely 'classical' considerations).

4. Free motion on Poisson $SU(N)$

The Poisson structure on $SL(N,\mathbb{C})$ viewed as the Heisenberg double of Poisson $SU(N)$ (analogue of the cotangent bundle) is given by

$$\{g_1, g_2\} = \rho g_1 g_2 + g_1 g_2 \rho \qquad \{g_1^\dagger, g_2\} = -g_1^\dagger w g_2 - g_2 w g_1^\dagger \tag{21}$$

(we have dropped the subscript '+', for simplicity).

In the non-deformed case of the cotangent bundle T^*G to $G = SU(N)$, the free motion is governed by the Hamiltonian $H : T^*G \to \mathbb{R}$ proportional to the square of the momentum, given by a bi-invariant metric on G. In other words, there is a distinguished quadratic function on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G (defined by the Killing form), and H is just the pullback of this function to T^*G (from the left or from the right—it does not matter since the quadratic function is invariant under coadjoint action of G; also: it is a Casimir for the Poisson structure on $\mathfrak{g}^*$).

When $T^*SU(N)$ is replaced by $SL(N,\mathbb{C})$, it is still easy to find a Hamiltonian with similar properties, namely

$$H(g) := \tfrac{1}{2}\,\mathrm{tr}\, g^\dagger g. \tag{22}$$

Note that it depends only on the 'momenta' $\beta \in G^*$:

$$H(g) = H(u\beta) = \tfrac{1}{2}\,\mathrm{tr}(u\beta)^\dagger u\beta = \tfrac{1}{2}\,\mathrm{tr}\,\beta^\dagger\beta = H(\beta).$$

It does not matter which way we decompose g:

$$H(g) = H(\beta' u') = \tfrac{1}{2}\,\mathrm{tr}(\beta' u')^\dagger \beta' u' = \tfrac{1}{2}\,\mathrm{tr}\,\beta'^\dagger\beta' = H(\beta')$$

which means that H as a function on G^* is invariant with respect to the dressing action:

$$H(\beta) = H(u\beta) = H({}^u\beta \cdot u^\beta) = H({}^u\beta)$$

(notation of [8, 15]). It means that H is a Casimir on G^*.

We shall now examine the equations of motion. We have

$$\begin{aligned}
\dot{g} = \{H, g\} &= \tfrac{1}{2}\,\mathrm{tr}_1\{g_1^\dagger g_1, g_2\} = \tfrac{1}{2}\,\mathrm{tr}_1(\{g_1^\dagger, g_2\}g_1 + g_1^\dagger\{g_1, g_2\}) \\
&= \tfrac{1}{2}\,\mathrm{tr}_1(-g_1^\dagger w g_2 g_1 - g_2 w g_1^\dagger g_1 + g_1^\dagger \rho g_1 g_2 + g_1^\dagger g_1 g_2 \rho) \\
&= -\tfrac{1}{2}\,\mathrm{tr}_1[g_1^\dagger(w-\rho)g_1 g_2 + g_1^\dagger g_1 g_2 (w-\rho)]
\end{aligned}$$

where tr_1 means the (partial) trace—with respect to the first indices only. Using (20) and the identity

$$\mathrm{tr}_1\, g_1^\dagger g_1 g_2 P = g g^\dagger g = \mathrm{tr}_1\, g_1 g_1^\dagger P g_2$$

(P denotes the permutation, as in (20)), we obtain

$$\dot{g} = \mathrm{i}\varepsilon\left[g g^\dagger g - \frac{1}{N}(\mathrm{tr}\, g^\dagger g) g\right]. \tag{23}$$

Substituting here $g = u\beta$, we obtain

$$\dot{u}\beta + u\dot{\beta} = \mathrm{i}\varepsilon\left[u\beta\beta^\dagger u^\dagger u\beta - \frac{1}{N}(\mathrm{tr}\,\beta^\dagger\beta) u\beta\right]$$

or,

$$u^{-1}\dot{u} + \dot{\beta}\beta^{-1} = \mathrm{i}\varepsilon\left[\beta\beta^\dagger - \frac{1}{N}(\mathrm{tr}\,\beta\beta^\dagger)\right].$$

Since the right-hand side belongs to $\mathfrak{g} = su(N)$, we have $\dot{\beta} = 0$, which was in fact also clear before, because H is a Casimir on G^*. Therefore we are left with the condition of constant velocity

$$u^{-1}\dot{u} = F(\beta) := \mathrm{i}\varepsilon\left[\beta\beta^\dagger - \frac{1}{N}(\mathrm{tr}\,\beta\beta^\dagger)\right]. \tag{24}$$

It follows that as far as configurations are concerned, the motion looks exactly as the non-deformed one: the particle moves on the 'big circles' (shifted one-parameter subgroups) with constant velocity. The difference consists of the momentum variables, which have a nonlinear nature. The function F above tells how to compute the velocity when the momentum is given. It plays the role of the inverse Legendre transformation.

A general notion of the Legendre transformation in the case of phase spaces of Poisson manifold is investigated in [19]. Here we shall show only two properties of the map $F : SB(N) \to su(N)$.

Proposition 4.1.

(1) F intertwines the dressing action with the adjoint action:

$$F({}^u\beta) = uF(\beta)u^{-1}.$$

(2) F is bijective.

Proof. The first property follows from the fact that if $u\beta = \beta' u'$, then $\beta'\beta'^{\dagger} = u\beta\beta^{\dagger}u^{\dagger}$. To prove the second, we first show that the map

$$SB(N) \ni \beta \mapsto \psi(\beta) = \beta\beta^{\dagger} \in P := \{p : p > 0, \det p = 1\}$$

is a bijection. Define a map $\phi : P \to SB(N)$ by

$$\phi(p) := \beta \qquad \text{where } \beta \text{ is such that } p^{\frac{1}{2}} = \beta u \qquad \text{(Iwasawa)}.$$

We have $\psi \circ \phi = \mathrm{id}$, since $\beta\beta^{\dagger} = (\beta u)(\beta u)^{\dagger} = p^{\frac{1}{2}}p^{\frac{1}{2}} = p$. But ϕ is also surjective, since, given $\beta \in SB(N)$, it is sufficient to consider its polar decomposition $\beta = p_0 u_0$ and notice that $\phi(p_0^2) = \beta$.

It remains to prove that the map

$$P \ni p \mapsto h = p - \frac{1}{N}\,\mathrm{tr}\, p \in \mathrm{i}\cdot su(N)$$

is a bijection. We first show that h determines p. Choose an orthonormal basis in which h is diagonal, then p is also diagonal in that basis. Let p_i and h_i be the corresponding eigenvalues, then

$$\lambda_i = p_i - \langle p\rangle \qquad \text{where } \langle p\rangle := \frac{1}{N}\sum_j p_j.$$

If λ_i come from some p, then

$$\lambda_i + \langle p\rangle > 0 \qquad (\lambda_1 + \langle p\rangle)\cdot\ldots\cdot(\lambda_N + \langle p\rangle) = 1.$$

Since the function

$$[\max_j(-\lambda_j), \infty[\ni t \mapsto f(t) := (\lambda_1 + t)\cdot\ldots\cdot(\lambda_N + t) \in [0, \infty[$$

is a (monotonic) bijection, there is exactly one t_0 such that $f(t_0) = 1$, hence $\langle p\rangle = t_0$ and this completely determines p by $p_i = \lambda_i + \langle p\rangle$. It is easy to see that $p_i = \lambda_i + t_0$, where $f(t_0) = 1$, defines some $p \in P$ for every h (because then $p_i > 0$ and $p_1\cdot\ldots\cdot p_n = 1$). $\square$

Finally, we remark that in the limit $\varepsilon \to 0$, the model becomes the undeformed one:

$$\beta \sim I + \varepsilon\xi \qquad \xi \in sb(N) \equiv su(N)^* \qquad F(\beta) \sim \mathrm{i}(\xi + \xi^{\dagger})$$

and

$$\frac{H(\beta) - \frac{1}{N}}{\varepsilon^2} \sim \frac{1}{2}\,\mathrm{tr}\,\xi\xi^{\dagger}.$$

In view of the existence of a Poisson isomorphism between $SB(N)$ and $su(N)^*$ [20], it would be interesting to find how the function H on $SB(N)$ is expressed as a function on $su(N)^*$.

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